Geometric Satake.

I. Geometry of affine Grassmannian.

Let G be a reductive group/ $\mathbb{C}$. Let $\mathcal{O} = \mathbb{C}[[z]]$, $K = \mathbb{C}((z))$. The affine Grassmannian is the Ind-scheme associated to the quotient $G(K)/G(\mathcal{O})$, denoted Gr_G. We will get more technical later.

Basic Example:

If $G = T$ is a torus, $T(K)/T(\mathcal{O})$ is identified w/ $X_*(T)$, the cocharacters of T: Gr_T is discrete. If $\lambda \in X_*(T)$, then z^λ is given by the unique elt of Gr_T s.t. $\mathrm{ord}_z(\chi(z^\lambda)) = \langle \chi, \lambda \rangle$ for all $\chi \in X^*(T)$.

Now back to general G.

$G(K) \circlearrowright Gr_G$ on the <u>left</u>.

Let $T \subset G$ be a maximal torus. $Gr_T \longrightarrow Gr_G$. Let $L_\lambda \in Gr_G$ be the image of each $\lambda \in X_*(T)$. Considering the $T(\mathbb{C}) \subseteq T(K) \subseteq G(K)$ action, these are precisely the $T(\mathbb{C})$ fixed points of Gr_G.

Also consider the $G_\mathcal{O} \subseteq G_K$ left action on Gr_G. The orbits are finite dimensional. They are parametrized by $X_*^+(T) = X_*(T)/W$; the orbit is precisely $\boxed{Gr^\lambda \triangleq G_\mathcal{O} L_\lambda}$, for $\lambda \in X_*^+$.

(Two λ in same W-orbit yield the same $G_\mathcal{O}$-orbit.)

Choose a Borel $B \supseteq T$, let N = unipotent radical of B. For dominant λ, μ we have:

$$\mathrm{Gr}^{\mu} \subseteq \overline{\mathrm{Gr}^{\lambda}} \quad \text{iff} \quad \lambda - \mu \text{ is a}$$

sum of positive coroots.

G, T, B refer to $G(\mathbb{C}) \subseteq G(K)$, $T(\mathbb{C}) \subseteq T(K)$, etc. The orbit $G \cdot L_\lambda \cong$ the flag manifold G/P_λ, where P_λ = stabilizer of L_λ in G = parabolic w/ Levi factor associated to $\{\alpha \in \Delta \mid \lambda(\alpha) = 0\}$. Now, we let $\check{\delta} : \mathbb{G}_m \longrightarrow \mathrm{Aut}\, \mathcal{K}$ be given by $\{s \longmapsto \{z \longmapsto sz\}\}$, $s \in \mathbb{C}^*$. Observe that:

$$\mathrm{Gr}^{\lambda} = \{x \in \mathrm{Gr} \mid \lim_{s \to 0} \check{\delta}(s)\, x \in G \cdot L_\lambda\}.$$

We see that Gr^{λ} is a G-equivariant vector bundle over G/P_λ. Say we pick a Borel $B \supseteq T$.

Let $\boxed{\rho = \text{the ubiquitous } 1/2 \text{ sum of pos. roots of } G.}$

If $\lambda \in X_*(T)$ is dominant, then:

$$\dim(\mathrm{Gr}^{\lambda}) = 2\rho(\lambda).$$

Let $ev : G_{\mathcal{O}} \longrightarrow G$, $z \longmapsto 0$. Let I = Iwahori $= ev^{-1}(B) \subseteq G_{\mathcal{O}}$; and let $\Gamma = ev^{-1}(1)$ be the highest congruence subgroup. The I-orbits are parametrized by $X_*(T)$; since I-orbits are also $ev^{-1}(N)$ orbits, they are affine spaces. Thus each $G_{\mathcal{O}}$ orbit acquires a cell-decomposition as union of I orbits. The Γ-orbit, $\Gamma \cdot L_\lambda$ is the fiber of the vector bundle: $\mathrm{Gr}^{\lambda} \longrightarrow G/P_\lambda$.

Let us define $G_{\mathcal{O}}^{-}$ to be the subgroup ind-scheme of $G_{\mathcal{K}}$ w/ $\mathbb{C}$-points $G(\mathbb{C}[z^{-1}])$. $G_{\mathcal{O}}^{-}$-orbits are <u>also</u> indexed by W-orbits in $X_*(T)$, given by $G_{\mathcal{O}}^{-} \cdot L_\lambda$, $\lambda \in X_*(T)$. These are "opposite" to $G_{\mathcal{O}}$-orbits:

$$G_{\mathcal{O}}^{-} \cdot L_\lambda = \{ x \in \mathrm{Gr} \mid \lim_{s \to \infty} \check{\delta}(s) x \in G \cdot L_\lambda \}.$$

which gives us:

$$\left(G_{\mathcal{O}}^{-} \cdot L_\lambda \right) \cap \overline{\mathrm{Gr}^\lambda} = G \cdot L_\lambda .$$

$G_{\mathcal{O}}^{-}$ contains $\Gamma^{-} = \ker\left(ev_\infty : z^{-1} \longmapsto 0 \right)$. The fiber of $G_{\mathcal{O}}^{-} \cdot L_\lambda \longrightarrow G/P_\lambda$ is $\Gamma^{-} \cdot L_\lambda$.

We now introduce the <u>semi-infinite orbits</u>:

For each $\nu \in X_*(T)$, let us consider

$\boxed{S_\nu \overset{\Delta}{=} N_{\mathcal{K}} L_\nu}$, where $N_{\mathcal{K}}$ = the ind-subscheme of G w/ $\mathbb{C}$-points $N(\mathcal{K})$. These have neither finite dimension nor codimension. They are ind-varieties: their intersection w/ any $\overline{\mathrm{Gr}^\lambda}$ is a bona-fide algebraic variety. Here are their basic properties:

Prop. (3.1 in MV).

a) $\overline{S_\nu} = \bigcup_{\eta \leq \nu} S_\eta$ where $\eta \leq \nu \Leftrightarrow \nu - \eta$ is a sum of positive roots,

b) Within $\overline{S_\nu}$, the boundary of S_ν is a hyperplane section under an embedding of Gr into projective space.

Now, we also have the "opposite" orbits. Let B^- be the opposite Borel; $\nu \in X_*(T)$.

Define: $\boxed{T_\nu \triangleq N_K^- \cdot L_\nu}$ $(N_K^- = R_u(B^-))$.

S_ν, T_ν intersect the Gr^λ as follows:

Thm. (3.2 in MV)

a) $S_\nu \cap Gr^\lambda$ is non-empty iff $L_\nu \in \overline{Gr^\lambda}$; in this case $S_\nu \cap \overline{Gr^\lambda}$ is pure of dimension $e(\nu+\lambda)$ (λ chosen dominant).

b) The intersection $T_\nu \cap Gr^\lambda$ is non-empty iff $L_\nu \in \overline{Gr^\lambda}$, in which case $T_\nu \cap \overline{Gr^\lambda}$ is pure of dimension $-e(\nu+\lambda)$, λ chosen anti-dominant.

Observation: $L_\nu \in \overline{Gr^\lambda}$ iff ν is a weight of the irreducible rep of $\check{G}_{\mathbb{C}}$ of highest weight λ; $\check{G}_{\mathbb{C}}$ the Langlands Dual group.

PF. Let's prove a); b) follows by symmetry. Let

$2\check{e} : \mathbb{G}_m \longrightarrow T$ = sum of positive coroots.

This acts by conjugation on N_K; observe that if $n \in N_K$, $\lim_{s\to 0} 2\check{e}(s)\, n = 1$. Thus any $x \in S_\nu$ satisfies $\lim_{s\to 0} 2\check{e}(s)\, x = L_\nu$. L_ν are fixed point for $\mathbb{G}_m$-action via $2\check{e}$; thus:

$$S_\nu = \{ x \in \mathcal{G}r \mid \lim_{s\to 0} 2\check{e}(s)\, x = L_\nu \}.$$

Thus: if $x \in S_\nu \cap \mathcal{G}r^\lambda$ since $\mathcal{G}r^\lambda$ is T-invariant. Thus $S_\nu \cap \mathcal{G}r^\lambda$ is nonempty iff $L_\nu \in \overline{\mathcal{G}r^\lambda}$, iff ν

is a weight of $V^\lambda \in \mathrm{Rep}(\check{G}_{\mathbb{C}})$. Now say ν is such a weight.

Let us study the extreme cases. We claim:

$$S_\nu \cap \overline{\mathcal{G}r^\nu} = N_{\mathcal{O}} \cdot L_\nu = \begin{cases} I \cdot L_\nu & \text{if } \nu \text{ is dominant} \\ \{L_\nu\} & \text{if } \nu \text{ is anti-dominant} \end{cases}$$

To see this, note $N_K = N_{\mathcal{O}} \cdot (N_K \cap \Gamma^-)$. So:

$$S_\nu \cap \overline{\mathcal{G}r^\nu} = N_{\mathcal{O}} \cdot (N_K \cap \Gamma^-) \cdot L_\nu \cap \overline{\mathcal{G}r^\nu} = N_{\mathcal{O}} \cdot \left((N_K \cap K_-) L_\nu \cap \overline{\mathcal{G}r^\nu} \right)$$

Now: $(N_K \cap \Gamma^-) \cdot L_\nu \subseteq \Gamma^- \cdot L_\nu$; and we know that $G_{\mathcal{O}}^- \cdot L_\nu \cap \overline{\mathcal{G}r^\nu} = G \cdot L_\nu$, so $\Gamma^- \cdot L_\nu \cap \overline{\mathcal{G}r^\lambda} = \{L_\nu\}$. This gives the first equality above. If ν is antidominant, then $N_{\mathcal{O}}$ stabilizes L_ν. If ν is dominant, $N_{\mathcal{O}}^-$ stabilizes $L_\nu \Rightarrow I \cdot L_\nu = B_{\mathcal{O}} N_{\mathcal{O}}^- \cdot L_\nu = B_{\mathcal{O}} \cdot L_\nu = N_{\mathcal{O}} \cdot L_\nu$.

Thus: thm. holds if $\nu = w_0 \cdot \lambda$ or $\nu = \lambda$ (w_0 = longest elt. of w). Now take arbitrary ν s.t. $L_\nu \in \overline{Gr^\lambda}$, $\nu > w_0 \cdot \lambda$, let C be irred component of $S_\nu \cap \overline{Gr^\lambda}$.

Let $C = C_0$, $d = \dim C$, H_ν = hyperplane of prev. prop.

Let D be an irred component of $\overline{C_0} \cap H_\nu$; $\dim D = d-1$,

$$D \subseteq \bigcup_{\mu < \nu} S_\mu.$$ Thus $\exists\ \nu_1 < \nu \stackrel{\circ}{=} \nu_0$ s.t. $C_1 = D \cap S_{\nu_1}$ is open, dense in D. $\dim C_1 = d-1$. Continue: we get a sequence of coweights ν_k, $k = 0, \dots, d$, $\nu_k < \nu_{k-1}$; a chain of irred components C_k of $S_{\nu_k} \cap \overline{Gr^\lambda}$ s.t. $\dim C_k = d-k$. Since C_d has $\dim = 0$, we see $\nu_d \geq w_0 \lambda$. Thus:

$$\dim C = d \leq e(\nu - w_0 \cdot \lambda).$$

Now we start from the opposite end: write $A_0 = S_\lambda \cap \overline{Gr^\lambda}$. Then $\overline{A}_0 = \overline{Gr^\lambda}$, $\dim A_0 = 2e(\lambda)$. As before, $C \subseteq \overline{Gr^\lambda} \Rightarrow$ $\exists$ component D of $\overline{A}_0 \cap H_\lambda$ s.t. $C \subseteq D$; hence $\exists\ \mu < \lambda$ and component A_1 of $S_\mu \cap \overline{Gr^\lambda}$ s.t. $\overline{A}_1 = D$, $\dim A_1 = 2e(\lambda) - 1$. We obtain a sequence of coweights μ_k, $k = 0, \dots, e$, w/ $\mu_0 = \lambda$, $\mu_e = \nu$, s.t. $\mu_k < \mu_{k-1}$; and a chain of irreducible components A_k of $S_{\nu_k} \cap \overline{Gr^\lambda}$ s.t. $\dim A_k = 2e(\lambda) - k$, $A_e = C$.

Thus: $\operatorname{codim}_{\overline{Gr^\lambda}} C = e \leq e(\lambda - \nu)$.

But $\dim C + \operatorname{codim}_{\overline{Gr^\lambda}} C = \dim \overline{Gr^\lambda} = 2e(\lambda)$.

So: $\dim C = e(\nu - w_0 \cdot \lambda)$;

$\operatorname{codim}_{\overline{Gr^\lambda}} = e(\lambda - \nu)$.

$\uparrow$

<u>Corollary.</u> For any dominant $\lambda \in X_*(T)$, any T-invariant closed subset $X \subseteq \overline{Gr^\lambda}$, we have

$$\dim(X) \leq \max_{L_\nu \in X^T} e(\lambda + \nu),$$

where $X^T = \{T\text{-fixed points of } X\}$.

<u>Pf.</u> $X \cap S_\nu$ is non-empty iff $L_\nu \in X$. since:

$$X = \bigcup_{L_\nu \in X^T} X \cap S_\nu \subseteq \bigcup_{L_\nu \in X^T} \overline{Gr^\lambda} \cap S_\nu,$$

apply previous theorem.

II. <u>The Category Sph and the Global Cohomology Functor.</u>

Slight subtlety in definition: want

category of perverse G_O-equivariant sheaves of Gr:

$$Sph \underset{def}{=} D_{G_O}(Gr) \underset{def}{=} \varinjlim_{\substack{\lambda \in X_*(T) \\ (\text{standard order})}} D_{G_O}(\overline{Gr^\lambda})$$

limit taken over closed embeddings.

Goal: Prove that

$Sph = D_{G_O}(Gr)$ and $Rep_{\check{G}_{\mathbb{C}}}$ are equivalent

as tensor categories.

We will do this via a Tannakian-formalism: show Sph is a semisimple Abelian category, and construct a faithful exact tensor functor:

$$Sph \longrightarrow Vect.$$

For formal reasons, then, our category will be equivalent to the category of representations of some (reductive!) group. It remains to verify that this has the combinatorics of the Langlands Dual. However, one can also use this Tannakian formalism as a natural way to define the Langlands Dual...

Observe:

$$\begin{tikzcd} & Gr_B \arrow[dl, "b"'] \arrow[dr, "t"] & \\ Gr_G & & Gr_T \end{tikzcd}$$

t preserves connected components; b is an injection of topological spaces. Observe that $S_\lambda = b[t^{-1}(t^\lambda)]$; at least set theoretically. Our Fiber Functor will be, w/ some modifications:

$$t_! b^* : Sph_G \longrightarrow Sph_T .$$

To make this precise, we return to the above geometry:

<u>Theorem</u>. (MV 3.5). $\forall\ \mathcal{A} \in Sph$, we have a canonical isomorphism:

$$H_c^k(S_\nu, \mathcal{A}) \longrightarrow H_{T_\nu}^k(Gr, \mathcal{A});$$

Moreover, both sides vanish, $k \neq 2e(\nu)$.

Thus, in particular: the Functors

$$F_\nu : Sph \longrightarrow Mod_{\mathbb{C}}$$

$$F_\nu \overset{\text{def}}{=} H_c^{2e(\nu)}(S_\nu -) = H_{T_\nu}^{2e(\nu)}(Gr, -)$$

are exact.

<u>Pf.</u> Let $\mathcal{A} \in Sph$, $\eta \in X_*(T)$ dominant. Then $\mathcal{A}|_{Gr^\eta}$ lies, as a complex of sheaves, in degrees $\leq -\dim(Gr^\eta) = -2e(\eta)$; i.e., $\mathcal{A}|_{Gr^\eta} \in D^{\leq -2e(\eta)}(Gr^\eta)$. Perversity $+$ $H_c^k(S_\nu \cap Gr^\eta) = 0$, for $k > 2\dim(S_\nu \cap Gr^\eta) = 2e(\nu+\eta)$ implies that

$$H_c^k(S_\nu \cap Gr^\eta, \mathcal{A}) = 0 \quad \text{if } k > 2e(\nu).$$

We may filter Gr by $\overline{Gr^{\eta}}$, so that $H_c^*(S_\nu, A)$ can be written in terms of $H_c^*(S_\nu \cap Gr^\nu)$; this gives us (via spectral sequence)

$$H_c^k(S_\nu, A) = 0 \quad \text{if} \quad k > 2e(\nu)$$

Analogously:

$$H_{T_\nu}^k(Gr, \mathcal{A}) = 0 \quad \text{if} \quad k < 2e(\nu).$$

Thus the isomorphism implies the theorem. Recall:

$\mathbb{G}_m \circlearrowright Gr$ via the cocharacter $2\check{e}$; fixed points are L_ν, $\nu \in X_*(T)$, and

$$S_\nu = \{x \in Gr \mid \lim_{s \to 0} 2\check{e}(s)x = L_\nu\}$$

$$T_\nu = \{x \in Gr \mid \lim_{s \to \infty} 2\check{e}(s)x = L_\nu\}.$$

The isomorphism is an on-the-nose consequence of Braden's Theorem.

<u>Thm.</u> We have a natural equivalence of functors:

$$H^* \cong F = \bigoplus_{\nu \in X_*(T)} H_c^{2e(\nu)}(S_\nu, -) : Sph \longrightarrow Vect.$$

Furthermore: F_ν and the equivalence are independent of choice of T and B.

<u>Pf.</u> Bruhat Decomposition of G_K w.r.t. Borels B_K, B_K^- $\Rightarrow$ decompositions $Gr = \bigcup S_\nu$; $Gr = \bigcup T_\nu$

$\Longrightarrow$ Two filtrations of Gr by $\overline{S_\nu}$ and $\overline{T_\nu}$.

$\Longrightarrow$ Two filtrations on cohomology H^*, both indexed by $X_*(T)$:

$$\ker : H^* \longrightarrow H^*_c(\overline{S_\nu}, -) \quad \text{and}$$

$$\operatorname{im} : H^*_{\overline{T_\nu}}(Gr, -) \longrightarrow H^*.$$

In degree $2e(\nu)$, we have:

$$H^{2e(\nu)}_{\overline{T_\nu}}(Gr, -) = H^{2e(\nu)}_{T_\nu}(Gr, -)$$

$$H^{2e(\nu)}_c(\overline{S_\nu}, -) = H^{2e(\nu)}_c(S_\nu, -)$$

and the composition:

$$H^{2e(\nu)}_{T_\nu}(Gr, -) \longrightarrow H^{2e(\nu)} \longrightarrow H^{2e(\nu)}_c(S_\nu, -)$$

is the above isomorphism.

Independence of T, B: Fix some $T_0 \subseteq B_0$ for reference, $\nu \in X_*(T_0)$, giving $S^0_\nu = (N_0)_K \cdot L_\nu$. G/T_0 parametrizes choice of pairs G/T_0. There is a canonical iso $T \longrightarrow T_0$; both are iso to the "universal" Cartan $B_0/N_0 = B/N$. Consider:

$$\begin{tikzcd}
Gr & Gr \times G/T_0 \arrow[l, "p"'] \arrow[d, "q"'] & S \arrow[l, "j"'] \arrow[d, "r"] \\
 & G/T_0 \arrow[r, equal] & G/T_0
\end{tikzcd}$$

where $S = \{(x, gT_0) \in Gr \times G/T_0 \mid x \in gS_\nu\}$; p, q, r are projections. For a point in G/T_0; i.e., a choice of $T \subseteq B$, the fiber of r is just the corr. set S_ν.

$\forall\ \mathcal{A} \in Sph$, we see that

$Rq_* j_! j^* p^* \mathcal{A}$ is a sub-local-system of

$Rq_* p^* \mathcal{A}$. But this is trivial.

Thus F_ν are independent of $T \subseteq B$.

Corollary. The Global cohomology functor

$$H^* = F : Sph \longrightarrow Vect \text{ is faithful, exact.}$$

Pf. Exactness follows from above Thm. Faithfulness:

Let $\mathcal{A} \in Sph$, $\exists$ Gr^λ open in $supp(\mathcal{A})$.

Choose λ dominant, then $T_\lambda \cap \overline{Gr^\lambda}$ is a point,

$F_\lambda(\mathcal{A}) \neq 0$. Since H^* does not annihilate nonzero objects

$\Rightarrow F_\nu$ is faithful.

III. The Convolution Product

Consider the diagrams:

$$Gr \times Gr \xleftarrow{p} G_K \times Gr \xrightarrow{q} G_K \times_{G_O} Gr \xrightarrow{m} Gr$$

$$(g g_O, g_O^{-1} g G_O)$$

p, q projection maps. Define

$$\mathcal{A}_1 * \mathcal{A}_2 = Rm_* \tilde{\mathcal{A}};$$

where $$q^* \tilde{\mathcal{A}} = p^*\left({}^pH^0(\mathcal{A}_1 \overset{L}{\boxtimes} \mathcal{A}_2)\right).$$

q^* is an iso between G_O-equivariant sheaves on $G_K \times Gr$ and sheaves on $G_K \times_{G_O} Gr$, since G_O action on $G_K \times Gr$ is free. And $p^*\left({}^pH^0(\mathcal{A}_1 \overset{L}{\boxtimes} \mathcal{A}_2)\right)$ is

$G_O \times G_O$ - equivariant.

Note: For $\mathbb{k}$ (cohomological coefficients) a field,

$\mathcal{A}_1 \overset{L}{\boxtimes} \mathcal{A}_2$ is perverse. MV prove that this still holds true over more general coefficient rings.

<u>Prop.</u> $\mathcal{A}_1 * \mathcal{A}_2$ is perverse if $\mathcal{A}_1, \mathcal{A}_2$ are.

<u>Pf.</u> MV introduce notion of stratified semi-small map. Say we have two stratified complex spaces:

$(Y, \mathcal{T})$, $(X, \mathcal{S})$, and a map $f: Y \longrightarrow X$.

Assume strata are locally trivial, w/ connected strata, f itself is stratified (i.e., if $T \in \mathcal{T}$, then $f(T)$ is a union of strata in $\mathcal{S}$, and $\forall S \in \mathcal{S}$ $f|_{f^{-1}(S)} : f^{-1}(S) \longrightarrow S$ is locally trivial in the stratified sense). Say f is stratified semi-small if

a) For any $T \in \mathcal{T}$, the map $f|_{\overline{T}}$ is proper

b) $\forall$ $T \in \mathcal{T}$, any $S \in \mathcal{S}$ s.t. $S \subseteq f(T)$, we have: $\dim\left(f^{-1}(x) \cap \overline{T}\right) \leq \frac{1}{2}\left(\dim f(\overline{T}) - \dim S\right)$ for any (thus all) $x \in S$.

A <u>Small</u> stratified map is $f: X \to Y$ s.t. $\exists$ nonempty $W \subset Y$, a dense stratified subset, s.t.

a) $\forall\ T \in \mathcal{T}$ the map $f|_{\overline{T}}$ is proper

b) the $f|_W : W \longrightarrow f(W)$ is finite and $W = f^{-1}(f(W))$.

c) $\forall\ T \in \mathcal{T}$, any $S \in \mathcal{S}$ s.t. $S \subseteq f(\overline{T}) - f(W)$, $\dim(f^{-1}(x) \cap \overline{T}) < \frac{1}{2}(\dim f(\overline{T}) - \dim S)$ for any (thus all) $x \in S$.

<u>Lemma.</u> If f is a stratified semismall map then $Rf_* \mathcal{A} \in \mathrm{Perv}_{\mathcal{S}}(X, \mathbb{K})$ $\forall$ $\mathcal{A} \in P_{\mathcal{T}}(Y, \mathbb{K})$. If f is a <u>small</u> stratified map then for any W as in the definition, and any $\mathcal{A} \in \mathrm{Perv}(W, \mathbb{K})$, we have $Rf_* j_{!*}\mathcal{A} = \tilde{j}_{!*} f_* \mathcal{A}$ for $j : W \hookrightarrow Y$, $\tilde{j} : f(W) \hookrightarrow X$ the two inclusions.

<u>Lemma.</u> The map $G_K \times_{G_O} \mathrm{Gr} \xrightarrow{m} \mathrm{Gr}$ is stratified semi-small, w.r.t. the Gr^λ-stratification.

<u>Pf.</u> Must verify: $\forall$ G_O-orbits Gr^ν in $\overline{\mathrm{Gr}^{\lambda+\mu}}$, the dimension $m^{-1}(L_\nu) \cap \widetilde{\mathrm{Gr}}^{\lambda,\mu}$ of $m : \widetilde{\mathrm{Gr}}^{\lambda,\mu} \longrightarrow \overline{\mathrm{Gr}^{\lambda+\mu}}$ at L_ν, is not more than $\frac{1}{2}\operatorname{codim}_{\overline{\mathrm{Gr}^{\lambda+\mu}}} \mathrm{Gr}_\nu$.

We may assume ν is anti-dominant, since $\mathrm{Gr}^{w \cdot \eta} = \mathrm{Gr}^\eta$, $w \in W$. For dominant η, $\dim \mathrm{Gr}^\eta = 2\rho(\eta)$ so the relevant codim is:

$$\operatorname{codim}_{\overline{\mathrm{Gr}^{\lambda+\mu}}} \mathrm{Gr}^\nu = 2\rho(\lambda+\mu) - 2\rho(w_0 \cdot \nu)$$
$$= 2\rho(\lambda + \mu + \nu).$$

It is incumbent upon us to prove:

$$\dim\left(m^{-1} L_\nu \cap \overline{\widetilde{Gr}^{\lambda,\mu}}\right) \le e(\lambda+\mu+\nu).$$

Let $p : G_K \times_{G_O} Gr \longrightarrow Gr$ via $(g, hG_O) \longmapsto gG_O$.

Then:

$$(p, m) : G_K \times_{G_O} Gr \xrightarrow{\sim} Gr \times Gr.$$

(p, m) carries the fiber $m^{-1} L_\nu$ to $Gr \times L_\nu$.

$p\left(m^{-1} L_\nu \cap \overline{\widetilde{Gr}^{\lambda,\mu}}\right)$ is T-invariant, whence we may apply our earlier results: we must find the T fixed-points in $p\left(m^{-1} L_\nu \cap \overline{\widetilde{Gr}^{\lambda,\mu}}\right) \subseteq \overline{Gr^\lambda}$.

These are precisely:

$$(z^\phi, z^\psi G_O) \text{ w/ } \phi, \psi \text{ weights}$$

of $L(\lambda)$ and $L(\mu)$ s.t. $\phi + \psi = \nu$.

I.e. ϕ, ψ are weights of irreps $L(\lambda')$, $L(\mu')$ w/ λ', μ' dominant, $\lambda' \le \lambda$, $\mu' \le \mu$, and $\phi + \psi = \nu$. But then:

$$\begin{aligned} e(\lambda+\phi) &\le e(\lambda+\phi) + e(\psi+\mu') \\ &= e(\lambda+\nu+\mu') \\ &\le e(\lambda+\nu+\mu). \end{aligned}$$

Thus:

$$\begin{aligned} \dim\left(p(m^{-1} L_\nu \cap \overline{\widetilde{Gr}^{\lambda,\mu}})\right) &\le \max_{L_\phi \in p(m^{-1} L_\nu \cap \overline{\widetilde{Gr}^{\lambda,\mu}})^T} e(\lambda+\phi) \\ &\le e(\lambda+\nu+\mu). \end{aligned}$$

IV. The Commutativity Constraint and the Fusion Product

Recall: Spherical Hecke Algebra is commutative.
We need the analogue here to show that

Idea of Drinfeld: interpret convolution as fusion.
We will now take a "global" (Beilinson–Drinfeld) version of our Affine Grassmannian.

Let X: smooth alg. curve
$x \in X \rightsquigarrow \mathcal{O}_x$, completed local ring; $K_x = \mathrm{Frac}(\mathcal{O}_x)$.

$\forall$ $\mathbb{C}$-algebras R, let $X_R = X \times \mathrm{Spec}(R)$,

$X_R^* = (X - \{x\}) \times \mathrm{Spec}(R)$.

Beauville–Laszlo Implies that $Gr_x = G_{K_x}/G_{\mathcal{O}_x}$ can be interpreted as the Ind-scheme representing

$$R \longmapsto \{ \mathcal{F} \text{ a } G\text{-torsor on } X_R,\ \nu: G \times X_R^* \longrightarrow \mathcal{F}|_{X_R^*} \text{ a trivialization on } X_R^* \}$$

$(\mathcal{F}, \nu)$ taken up to isomorphism.

We may perform this construction w/ arbitrary #'s of points: let $X^n = X \times \cdots \times X$ and consider the functor:

$$R \longmapsto \left\{ \begin{array}{l} (x_1, \ldots, x_n) \in X^n(R),\ \mathcal{F} \text{ a } G\text{-torsor on } X_R, \\ \nu_{(x_1, \ldots, x_n)} \text{ a trivialization of } \mathcal{F} \text{ on } X_R - \Gamma_{x_i} \end{array} \right\}$$

where $x_i : \mathrm{Spec}\, R \longrightarrow X$ and $\Gamma_{x_i} \subseteq X \times \mathrm{Spec}\, R$ is the graph of x_i. Call the ind-scheme represented by this functor: Gr_{X^n}.

Observe: $Gr_{X^n} \longrightarrow X^n$ w/ fiber $\prod_{i=1}^{k} Gr_{y_i}$ where

$\{y_1, \dots, y_k\} = \{x_1, \dots, x_n\}$ w/ all y_i distinct.

Now we upgrade our convolution product:

$$Gr_X \times Gr_X \xleftarrow{p} \widetilde{Gr_X \times Gr_X} \xrightarrow{q} Gr_X \tilde{\times} Gr_X \xrightarrow{m} Gr_{X^2} \xrightarrow{\pi} X^2$$

where $\widetilde{Gr_X \times Gr_X}$ denotes the ind-scheme:

$$R \longmapsto \left\{ \begin{array}{l} (x_1, x_2) \in X^2(R),\ \mathcal{F}_1, \mathcal{F}_2 \text{ } G\text{-torsors on } X_R;\ \nu_i \text{ a trivialization} \\ \text{of } \mathcal{F}_i \text{ on } X_R - \Gamma_{x_i} \quad (i=1,2);\ \mu_1 \text{ a trivialization} \\ \text{of } \mathcal{F}_1 \text{ on } \widehat{(X_R)}_{x_2} \end{array} \right\}$$

$\widehat{(X_R)}_{x_2}$ is the formal nbhd of x_2 in X_R. The "twisted" product

$Gr_X \tilde{\times} Gr_X$ is the ind-scheme representing

$$R \longmapsto \left\{ \begin{array}{l} (x_1, x_2) \in X^2(R);\ \mathcal{F}_1, \mathcal{F} \text{ } G\text{-torsors on } X_R;\ \nu_1 \text{ a trivialization} \\ \text{of } \mathcal{F}_1 \text{ on } X_R - x_1;\ \eta : \mathcal{F}_1\big|_{(X_R - \Gamma_{x_1})} \xrightarrow{\sim} \mathcal{F}\big|_{(X_R - \Gamma_{x_1})} \end{array} \right\}$$

We define p, q, m: Observe that all of these maps are "over" X^2 so we omit the point (x_1, x_2) from each:

p: Forget μ_1

q: $(\mathcal{F}_1, \nu_1, \mu_1;\ \mathcal{F}_2, \nu_2) \longmapsto (\mathcal{F}_1, \nu_1, \mathcal{F}, \eta)$

where $\mathcal{F}$ is the G-torsor obtained by gluing $\mathcal{F}_1$ on $X_R - \Gamma_{x_2}$ and $\mathcal{F}_2$ on $\widehat{(X_R)}_{x_2}$ using the iso induced by $\nu_2 \circ \mu_1^{-1} : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ on

$(X_R - \Gamma_{x_2}) \cap \widehat{(X_R)}_{x_2}$ (which is an R-family of punctured discs).

$$m : (\mathcal{F}_1, \nu_1, \mathcal{F}, \eta) \longmapsto (\mathcal{F}, \nu)$$

where $\nu = (\eta \circ \nu_1)\big|_{X_R - \Gamma_{x_1} - \Gamma_{x_2}}$.

What is the global analogue of G_O?

We call it $G_{X^n, O}$ which represents:

$$R \longmapsto \left\{ \begin{array}{l} (x_1, \dots x_n) \in X^n(R), \quad \mathcal{F} \text{ the trivial } G\text{-torsor on } X_R, \\ \mu_{(x_1, \dots, x_n)} \text{ a trivialization of } \mathcal{F} \text{ on } \widehat{(X_R)}_{(\Gamma_{x_1} \cup \dots \cup \Gamma_{x_n})} \end{array} \right\}$$

Now we define the convolution product of $\mathcal{B}_1, \mathcal{B}_2 \in \mathrm{Perv}_{G_{X,O}}(Gr_X, k)$ by:

$$\mathcal{B}_1 \underset{X}{*} \mathcal{B}_2 = Rm_* \tilde{\mathcal{B}}$$

where $$q^* \tilde{\mathcal{B}} = p^* ({}^p H^0 (\mathcal{B}_1 \overset{L}{\boxtimes} \mathcal{B}_2)).$$

But how does $G_{X,O}$ act on all the spaces?

Firstly: Gr_{X^2} can also be written as:

$$R \longmapsto \left\{ \begin{array}{l} x \in X(R), \quad \mathcal{F} \text{ a } G\text{-torsor on } \widehat{(X_R)}_x \\ \nu_x \text{ a trivialization of } \mathcal{F} \text{ on } \widehat{(X_R)}_x - \Gamma_x \end{array} \right\}.$$

$G_{X,O} \circlearrowright Gr_X$ by altering the trivialization ν_x; hence we may define $\mathrm{Perv}_{G_{X,O}}(Gr_X, k)$.

On $\widetilde{Gr_X \times Gr_X}$ there are <u>two</u> relevant $G_{X,O}$ actions.

1) view $G_{x,0}$ as a group scheme $/X^2$ via pulling it back along second projection

Then $G_{x,0} \circlearrowright \widetilde{Gr_X \times Gr_X}$ via altering μ_1.

2) write $\widetilde{Gr_X \times Gr_X}$ as:

$$R \longmapsto \left\{ \begin{array}{l} (x_1, x_2) \in X^2(R);\ \text{for } i=1,2\ \mathcal{F}_i \text{ is a } G\text{-torsor on } \widehat{(X_R)}_{x_i} \\ \nu_i \text{ is a trivialization of } \mathcal{F}_i \text{ on } \widehat{(X_R)}_{x_i} - \Gamma_{x_i}, \\ \mu_1 \text{ is a trivialization of } \mathcal{F}_1 \text{ on } \widehat{(X_R)}_{x_2} \end{array} \right\}$$

Again view $G_{x,0}$ as a group scheme on X^2 via pullback along second projection; $G_{x,0}$ alters both μ_1 and ν_2. This action is free, exhibiting

$$q: \widetilde{Gr_X \times Gr_X} \longrightarrow Gr_X \tilde{\times} Gr_X$$

as a $G_{x,0}$-torsor.

So $\widetilde{B}$ defined above exists and is unique.

Now: observe m is a stratified small map.

Indeed: let $\Delta \subset X^2$ be the diagonal; set $U = X^2 - \Delta$.

Then let W be the locus of points lying over U. Over U, m is an isomorphism; over Δ the map m is as in the previous section (which is semismall).

Now we construct the commutativity constraint. To simplify let $X = \mathbb{A}^1$, giving us a global coordinate. This trivializes Gr_X over X.

Write $\tau: Gr_X \longrightarrow Gr$ for the projection.

Restrict Gr_{X^2} to the diagonal $\Delta \cong X$, and to U. These restrictions are isomorphic to

Gr_X and to $(Gr_X \times Gr_X)|_U$ respectively.

[note: this appears to violate semi-continuity! This is because we are in the infinite dimensional setting. See Zhu's article for a discussion of this phenomenon.]

We get:

$$\begin{tikzcd}
Gr_X \arrow[r, "i"] \arrow[d] & Gr_{X^2} \arrow[d] & (Gr_X \times Gr_X)|_U \arrow[l, "j"'] \arrow[d] \\
X \arrow[r, "\Delta"] & X^2 & U \arrow[l, hook']
\end{tikzcd}$$

Let us write $\tau^\circ = \tau^*[1]$ (truncation functor):

$$\mathrm{Perv}_{G_O}(Gr, k) \longrightarrow \mathrm{Per}_{G_{X,O}}(Gr_X, k);$$

write $i^\circ = i^*[-1] : \mathrm{Perv}_{G_{X,O}}(Gr_X, k) \longrightarrow \mathrm{Perv}_{G_O}(Gr, k)$.

For $A_1, A_2 \in \mathrm{Perv}_{G_O}(Gr, k)$ we have:

a) $\tau^\circ A_1 \underset{X}{*} \tau^\circ A_2 \cong j_{!*}\left({}^pH^\circ(\tau^\circ A_1 \overset{L}{\boxtimes} \tau^\circ A_2)|_U\right)$

b) $\tau^\circ (A_1 * A_2) \cong i^\circ(\tau^\circ A_1 \underset{X}{*} \tau^\circ A_2)$

a) follows from smallness of m;

b) follows from definitions.

The payoff:

$$\tau^\circ(A_1 * A_2) \cong i^\circ j_{!*}\left({}^pH^\circ(\tau^\circ A_1 \overset{L}{\boxtimes} \tau^\circ A_2)|_U\right)$$

$$\cong i^\circ j_{!*}\left({}^pH^\circ(\tau^\circ A_2 \overset{L}{\boxtimes} \tau^\circ A_1)|_U\right) \cong \tau^\circ(A_2 * A_1)$$

Specializing this to any point on the diagonal gives a functorial iso from $A_1 * A_2$ to $A_2 * A_1$. This gives us a commutativity constraint.

V. Finishing the Argument.

i) We actually have to modify the commutativity constraint by a sign convention to give us the correct identification $\mathrm{Sph} \longrightarrow \mathrm{Rep}(\bar{G})$. This is analogous to the $|\text{modular character}|^{1/2}$ in the usual Satake transform; we require it above because of our artificial choice of coordinate in $\mathbb{A}^1$.

ii) The magic of Tannakian formalism gives us the existence of $\underline{a}$ group (which we can see is reductive) so we need only identify the root systems. This is done thru engaging directly w/ the combinatorics.